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A Combinatorial Model for Determining Penetration Probability of Warheads Accompanied by Decoys

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FOREWORD

This note presents the derivation, using the rudiments of probability and combinatorial techniques, of a mathematical model for evaluating the penetration probability of warheads accompanied by indistinguishable decoys. The resulting model is to be used in examining the utility of multiple warheads and decoys in Study 17.1, "Offensive Missiles Systems for the Field Army (U)." Because of its usefulness in the warhead-decoys problem, it was felt the description of the model should appear as an unclassified note independent of the report of the general problem of Study 17.1.

Appendix B includes a FORTRAN program of the model, written by Mrs. Bertha M. Butler and Mrs. Miriam K. Manoff, used to compute the data in the examples given in the paper.

The author is grateful to his colleague Dr. W. Bruce Taylor for his valuable assistance in establishing some of the preliminary results for this paper; to Walter Eckhardt for his helpful suggestions in revising the paper; to Charles R. Wyman for his support; and to Robert G. Busacker for his review of the final draft and useful observations.

CONTENTS

SUMMARY

Problem

To determine the probability that at least u of w warheads accompanied by d decoys penetrate to a target defended by an antimissile missile (AMM) system of c missiles, each with a single-shot kill probability (SSKP) p_m.

Facts

There are many techniques, known as "penetration aids," to counter an AMM system. One such technique is the use of multiple warheads and decoys. Forcing the defense to commit some of his interceptors to decoys, and/or forcing him to dilute his fire among several incoming warheads, results in a corresponding increase in penetration probability of the warheads.

This penetration probability is needed to evaluate the effectiveness of the use of multiple warheads and decoys.

Discussion

The problem is approached by using the rudiments of probability and combinatorial techniques. With these notions the problem is reduced to a version of a hypergeometric distribution.

The following assumptions are made ~~in the analysis~~: (a) the defense detects the complete swarm of incoming objects before the engagement begins; (b) he commits his c missiles as uniformly as possible; (c) he cannot distinguish between decoys and warheads prior to final commitment time; and (d) he cannot kill more than one incoming object with a single missile.

In the derivation of the mathematical model, a restriction of the number of incoming objects is first made, and the probability of at least one warhead penetrating is determined. Next this restriction is removed, and the probability of at least one warhead penetrating is established again. Finally this result is made more general to determine the probability that at least u warheads penetrate.

The paper results in a mathematical model for determining penetration probabilities of warheads accompanied by decoys. Two illustrations of the use of the model are presented.

A Combinatorial Model for Determining Penetration Probability of Warheads Accompanied by Decoys

GLOSSARY

AMM	antimissile missile
SSKP	single-shot kill probability
c	simultaneous intercept capacity of AMM system
p_m	SSKP of AMM
p_{2m}	SSKP of two AMMs
p_{im}	SSKP of i independently launched AMMs, i.e., the probability that at least one of the i missiles aimed at an object will destroy it = $1-(1-p_m)^i$
$P(u)$	probability that at least u warheads penetrate
$P[u]$	probability that exactly u warheads penetrate
d	number of decoys arriving over target
w	number of warheads arriving over target
$W_{[k]}$	probability that exactly k warheads are attacked by AMMs with SSKP = p_m
$W[i,j,k]$	probability that exactly i warheads are attacked by AMMs with SSKP = p_{im}, j of these are destroyed, and k other warheads are attacked and destroyed by AMMs with SSKP = $p_{(i+1)m}$

INTRODUCTION

During the study of penetration methods for field army missiles the use of decoys with warheads in a number of combinations was considered. These decoys could be launched in combination with the warhead in the same missile or separately as dummy missiles. In addition a multiple warhead could be delivered by one missile or single warheads by single missiles. A measure of effectiveness of this swarm of objects (warheads and decoys) arriving over a target was needed.

This paper develops a mathematical model for measuring this effectiveness by computing the penetration probability of warheads accompanied by indistinguishable decoys. The parameters of the model include the number of warheads and decoys arriving over the target, the capacity of the AMM defense, and the SSKP of each AMM. The paper concludes with two illustrations of the use of the model.

STATEMENT AND SCOPE OF THE PROBLEM

The offense has three modes of attack in attempting to penetrate an AMM system. First, he can stagger his warheads and decoys in time so that the defense will not be able to commit his missiles efficiently. The defense will be forced to decide arbitrarily to commit a number of AMMs to each of the various waves until his missile supply is exhausted.

A second mode, somewhat similar to the first, is a shoot-look-shoot or repeated attack. The offense attacks the missile supply of the defense but does not risk wasting warheads in overkilling. Like the staggered attack, this mode forces the defense to fire in ignorance of the number of incoming objects, so that he cannot use his missiles efficiently. Denying information to the defense concerning the number of incoming objects is excellent strategy for attacking his missile supply.

The third mode (and the one treated in this paper) is the simultaneous attack in which the defense knows the number of incoming objects and therefore can commit his simultaneous intercept capacity uniformly and efficiently. Uniform fire doctrine is discussed in App A. It is assumed that the offense has dispersed its objects in space so that only one can be destroyed by a single AMM. It is also assumed that the decoys cannot be distinguished from the warheads prior to last commitment time.

In this type of engagement the penetration probability depends on the number of incoming warheads and decoys, the SSKP of each AMM, and the simultaneous intercept capacity of the defensive system. Since no missile can be 100 percent reliable, this SSKP can be adjusted to include the degree

of reliability of the AMM. The capacity of the defense is interpreted as the number of AMMs launched against a single attack of incoming objects.

In the derivation of the model an initial restriction on the number of incoming objects is imposed. The number of objects in the swarm is no more than the defense capacity nor less than one-half this capacity. The probability of at least one warhead penetrating is established with this restriction. The model is then made more general by relaxing the restriction on the number of objects in the swarm. And finally the model is generalized to determine the probability of penetration by any number of warheads.

The problem stated precisely is: Given that d decoys and w warheads arrive over a target defended by an AMM system with capacity c missiles, each with SSKP $= p_m$, find the probability that at least u of the warheads penetrate this defense.

DERIVATION OF THE MODEL

Let it be assumed that $w+d \le c$ and $2(w+d) \ge c$, or equivalently let c be on the interval

$$c \begin{cases} \le 2(w+d) \\ \ge (w+d) \end{cases} \tag{1}$$

Restriction 1 ensures that no more than two nor less than one AMM can be committed to each incoming object. The number of incoming objects attacked by two missiles is

$$y = c-(w+d) \tag{2}$$

and the number of incoming objects attacked by one missile is

$$x = w+d-y = 2(w+d)-c \tag{3}$$

If the SSKP of a single AMM is p_m, then each of the incoming objects attacked by two missiles can be viewed as being attacked by a "new" single AMM with increased kill probability given by

$$p_{2m} = 1-(1-p_m)^2 \tag{4}$$

These "new" missiles will be identified as AMMs with SSKP $= p_{2m}$.

The probability that at least one warhead penetrates equals the complement of the probability that none of the warheads penetrate or

$$P(1) = 1-P[0] \tag{5}$$

So that none of the warheads may penetrate, each must be successfully intercepted by either an AMM with SSKP $= p_m$ or an AMM with SSKP $= p_{2m}$. Suppose exactly k warheads are intercepted by AMMs with SSKP $= p_m$. These k warheads can be selected in

$$\binom{w}{k} = \frac{w!}{k!\,(w-k)!} \tag{6}$$

ways. After k AMMs with SSKP = p_m are assigned to these k warheads, it follows from Eq 3 that there are $(x-k)$ AMMs remaining to be assigned to the decoys. This can be done in

$$\binom{d}{x-k} = \frac{d!}{(x-k)!\,(d-x+k)!} \tag{7}$$

ways.

The number of ways of assigning AMMs with SSKP = p_m to exactly k warheads is therefore the product $\binom{w}{k}\binom{d}{x-k}$. The total number of ways of assigning these AMMs with SSKP = p_m to all incoming objects is $\binom{w+d}{x}$, so that the probability that exactly k warheads are attacked by AMMs with SSKP = p_m is

$$W_{[k]} = \frac{\binom{w}{k}\binom{d}{x-k}}{\binom{w+d}{x}} \tag{8}$$

The system of probabilities defined by Eq 8 is called the "hypergeometric distribution." It should be noted that the probabilities $W_{[k]}$ are defined only for k not exceeding w or x. However, by employing the usual convention, $\binom{a}{b} = 0$ whenever $b > a$, then $W_{[k]} = 0$ whenever $k > w$ or $k > x$. Accordingly Eq 8 may be used for all $k \geq 0$ provided that the relation $W_{[k]} = 0$ is interpreted as impossibility.

The probability that the $(w-k)$ remaining warheads are attacked by AMMs having SSKP = p_{2m} is unity, since restriction 1 ensures that each one is attacked by at least one AMM. If exactly k warheads are attacked by AMMs having SSKP = p_m, then the remaining warheads are attacked by the "new" missiles.

Based on the assumption that all intercepts are independent, the probability that all k warheads attacked by AMMs with SSKP = p_m are destroyed is p_m^k. The remaining $(w-k)$ warheads are destroyed with the probability p_{2m}^{w-k}.

The conditional probability that exactly k warheads are attacked and destroyed by AMMs with SSKP = p_m and the remaining warheads are destroyed by AMMs with SSKP = p_{2m} is given by

$$W[k,k,w-k] = \frac{\binom{w}{k}\binom{d}{x-k}}{\binom{w+d}{x}} p_m^k\, p_{2m}^{w-k} \tag{9}$$

And finally this number of k warheads can range from 0 to w, and the corresponding values of Eq 9 are the probabilities that no warhead penetrates.

These events are mutually exclusive and exhaustive so that the probability that all warheads are destroyed, i.e., the probability that no warhead penetrates, is obtained by summing Eq 9 over all values of k. This probability is given by

$$P[0] = \frac{1}{\binom{w+d}{x}} \sum_{k=0}^{w} \binom{w}{k} \binom{d}{x-k} p_m^k \, p_{2m}^{w-k} \tag{10}$$

Using Eqs 3 and 5, the probability that at least one warhead penetrates is obtained and given by

$$P(1) = 1 - \frac{1}{\binom{w+d}{2(w+d)-c}} \sum_{k=0}^{w} \binom{w}{k} \binom{d}{2(w+d)-c-k} p_m^k \, p_{2m}^{w-k} \tag{11}$$

It should be noted that Eq 11 is meaningless when the value of c violates restriction 1. The factor $\binom{w+d}{2(w+d)-c}$ in the denominator of Eq 11 becomes zero whenever $c > 2(w+d)$ or $c < (w+d)$. In order to extend the use of Eq 11 for values of c outside the interval 1 the auxiliary relations

$$i = \left[\frac{c}{w+d}\right] \tag{12}$$

$$c_i = c - (i-1)(w+d) \tag{13}$$

$$p_{im} = 1 - (1-p_m)^i \tag{14}$$

$$p_{(i+1)m} = 1 - (1-p_m)^{i+1} \tag{15}$$

are introduced.

In the above relations the number i is the minimum number of AMMs that attack any one object and is determined as the integral part of the quotient $c/(w+d)$; c_i can be regarded as a pseudocapacity of AMMs with kill probabilities p_{im} and $p_{(i+1)m}$. For $i = 1$ the case just discussed is obtained, i.e., all incoming objects are attacked by at least one AMM, and some are attacked by two AMMs.

Eq 11 can be made more general by considering the interval

$$c_i \begin{cases} \leq (i+1)(w+d) \\ \geq i(w+d) \end{cases} \tag{16}$$

where i is given by Eq 12. For $i = 1$ this interval reduces to the initial restriction 1. The number of incoming objects attacked by i AMMs becomes

$$x_i = 2(w+d) - c_i \tag{17}$$

The probability that a warhead is destroyed by the i AMMs assigned to it is given by Eq 14. The generalization of Eq 11 continues by analogy resulting in the probability that none of the warheads penetrate, given by

$$P[0] = \frac{1}{\binom{w+d}{x_i}} \sum_{k=0}^{w} \binom{w}{k} \binom{d}{x_i - k} p_{im}^{k} \, p_{(i+1)m}^{w-k} \tag{18}$$

Finally from Eq 5 the probability that at least one warhead penetrates is given by

$$P(1) = 1 - \frac{1}{\binom{w+d}{x_i}} \sum_{k=0}^{w} \binom{w}{k} \binom{d}{x_i - k} p_{im}^{k} \, p_{(i+1)m}^{w-k} \tag{19}$$

For the special case when $i = 0$, i.e., when the number of incoming objects equals or exceeds the defense capacity, Eq 19 reduces to

$$P(1) = 1 - \frac{\binom{d}{w+d-c}}{\binom{w+d}{c}} p_m^{w} \tag{20}$$

It can be observed from Eq 20 that if $w > c$, then obviously the probability of penetration of at least one warhead is unity.

Another specialization of Eq 19 of interest is for $d = 0$, i.e., when no decoys are used. Equation 19 then reduces to

$$P(1) = 1 - p_{im}^{2w-c_i} \, p_{(i+1)m}^{c_i - w} \tag{21}$$

GENERALIZATION

It is now easy to generalize Eq 19 to obtain the probability that at least u warheads penetrate. First, it is necessary to obtain the probability that exactly u warheads penetrate. As before, the warheads can be destroyed either by an AMM with SSKP = p_{im} or by one with SSKP = $p_{(i+1)m}$. The probability that exactly u warheads penetrate is the product of (a) the probability that exactly r warheads are attacked by AMMs with SSKP = p_{im} times (b) the probability that j of these warheads are destroyed times (c) the probability that the remaining $(w-r)$ warheads are attacked by AMMs with SSKP = $p_{(i+1)m}$ times (d) the probability that exactly $(w-u-j)$ of these warheads are destroyed. The product of b and d must be summed from $j = 0$ to $j = r$, c equals unity, and a must be summed from $r = 0$ to $r = w$. The resulting probability that exactly u warheads penetrate is given by

$$P[u] = \sum_{r=0}^{w} \frac{\binom{w}{r}\binom{d}{x_i - r}}{\binom{w+d}{x_i}} \sum_{j=0}^{r} \binom{r}{j} p_{im}^{j} (1 - p_{im})^{r-j} \binom{w-r}{w-u-j} p_{(i+1)m}^{w-u-j} \left(1 - p_{(i+1)m}\right)^{u-r+j} \tag{22}$$

Finally the probability that at least u warheads penetrate is

$$P(u) = \sum_{k=u}^{w} P[k] \tag{23}$$

or

$$P(u) = 1 - \sum_{k=0}^{u-1} P[k] \tag{24}$$

Equation 24 reduces to Eq 19 for $u = 1$.

USE OF THE MODEL

This final section presents two examples illustrating the use of the model. An obvious example is that of determining the warhead-decoy requirement for penetrating a given defense. In this example the increased payoff obtained when using two warheads instead of one warhead per target is clearly revealed.

The second example points out a method for the offense to maximize the number of targets destroyed by offensive warheads. Also clearly illustrated is the enhancement of the penetration probability by the use of decoys.

Example I

Number of Decoys Required. Suppose the offense desires a 90 percent assurance of penetrating the defense. In this instance penetration means one or more warheads surviving the AMM attack. Let the defense have a capacity of six AMMs, each with SSKP = .75. The number of decoys required using w warheads can be calculated using Eq 20. Table 1 lists the results of this calculation.

TABLE 1

RELATION BETWEEN WARHEADS AND DECOYS NEEDED FOR 90 PERCENT ASSURANCE OF PENETRATING DEFENSE CAPACITY OF SIX AMMs WITH SSKP = .75

Warheads w	Decoys d
1	44
2	11
3	6
4	3
5	2

In this example, the offense can penetrate the defense with certainty using seven warheads and no decoys. From Table 1 it is evident that the offense can attain 90 percent probability of penetration with fewer warheads if decoys are used. It is interesting to observe that as the offense's choice of the number of warheads decreases from 3 to 2 to 1, the corresponding number of decoys required for 90 percent penetration probability increases from 6 to 11 to 44. In

like manner the number of decoys required for penetration against defenses of other capacities has been computed for a range of defenses with $p_m = .75$. The results are shown in Fig. 1.

Figure 2 demonstrates the relation of required decoys to penetration probability. This figure again illustrates the increased payoff secured by the use of two warheads. With one warhead the number of decoys required to obtain a spread of from 50 to 90 percent probability of penetration ranges from 8 to 44. With two warheads a range from 4 to 11 decoys yields the same spread in penetration probability.

Example II

Optimum Attack. Suppose the offense wishes to destroy a large number of targets. Let each of these targets be defended by an AMM system with capacity 20 missiles, each with SSKP = .75. The problem to be solved here is how the offense should allocate its warheads to this set of targets. It is true that if the offense elects to send 21 warheads against a single target at least one warhead will certainly penetrate. However, the offense can employ a smaller number of warheads per target and attack a larger number of targets, but the probability of penetration decreases for each target. The important result is that this tactic yields a larger number of targets destroyed, up to a point. After this point has been reached, the use of fewer warheads per target results in a decreasing number of targets destroyed. In other words the optimum attack maximizes the number of targets killed or equivalently minimizes the average number of warheads required to destroy one target.

As an example, by letting $d = 0$ in Eq 19 the penetration probability using w warheads against a defense of 20 missiles each with SSKP = .75 can be computed. The results are displayed in Table 2. In this example the minimum number of warheads per target killed on a statistical basis occurs at $w = 13$. Because the number of warheads per target killed is an average, it need not be an integer. It should be noted also that the probability of penetration corresponding to this minimum point is relatively high, about 89 percent.

The penetration probability can be increased further by adding decoys. Tables 3 and 4 are constructed in a manner similar to that used for Table 2 but with $d = 3$ and $d = 6$ in Eq 19. In Table 4 note that the probability of penetration using 13 warheads has now increased to 97 percent.

The offense wants to destroy a target for each investment of w warheads per target. This relation is plotted as a straight line in Fig. 3. Also displayed in Fig. 3 are decoy curves asymptotic to this straight line. After the offense decides on w (warheads per target he will use), he must select a decoy curve that closely approximates the straight line beginning at w. This number of decoys will obtain the maximum number of targets killed.

The effect of adding decoys can be further illustrated by plotting w warheads as a function of probability of penetration. This plot results in an S-shaped or logistic curve. The penetration probability rises slowly at first, then rapidly with the addition of more warheads, and finally a point of diminishing returns is reached. With the addition of decoys this logistic curve is horizontally translated to the left as shown in Fig. 4.

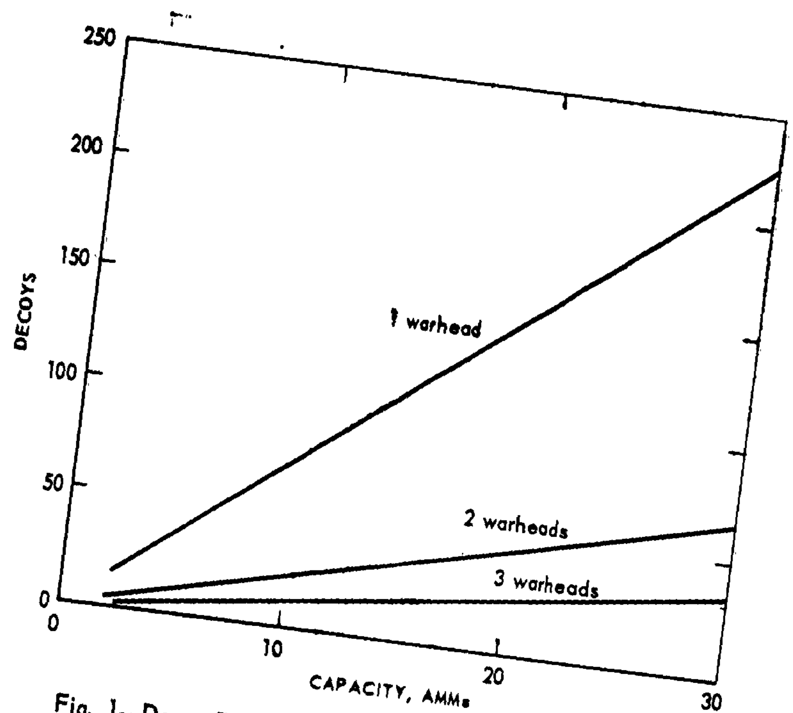

Fig. 1—Decoy Requirement as a Function of Capacity
$P(1) = .90$; $p_m = .75$

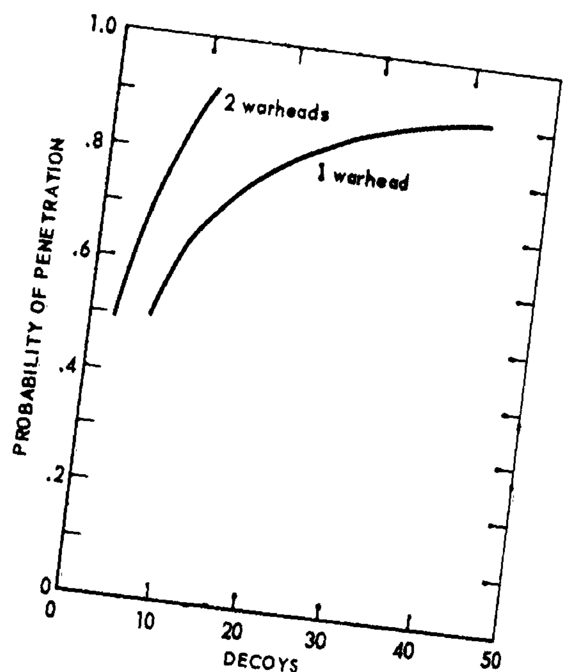

Fig. 2—Decoy Requirement as a Function of Penetration Probability
$p_m = .75$; $c = 6$

TABLE 2

PROBABILITY OF PENETRATION USING w WARHEADS, NO DECOYS, AGAINST A DEFENSE WITH CAPACITY 20 AND SSKP = .75

Warheads w	Probability of penetration P(1)	w/P(1)
7	.147	47.6
8	.275	29.1
9	.383	23.5
10	.475	21.0
11	.685	16.1
12	.811	14.8
13	.887	14.7
14	.932	15.0
15	.959	15.6
16	.975	16.4
17	.985	17.3
18	.991	18.1

TABLE 3

PROBABILITY OF PENETRATION USING w WARHEADS, 3 DECOYS, AGAINST A DEFENSE WITH CAPACITY 20 AND SSKP = .75

Warheads w	Probability of penetration P(1)	w/P(1)
5	.181	27.6
6	.275	21.8
7	.363	19.3
8	.565	14.2
9	.709	12.7
10	.809	12.3
11	.877	12.5
12	.921	13.0
13	.950	13.7
14	.969	14.5
15	.980	15.3
16	.987	16.2
17	.992	17.1

TABLE 4

PROBABILITY OF PENETRATION USING w WARHEADS, 6 DECOYS, AGAINST A DEFENSE WITH CAPACITY 20 AND SSKP = .75

Warheads w	Probability of penetration P(1)	w/P(1)
3	.148	20.2
4	.227	17.6
5	.402	12.4
6	.557	10.8
7	.684	10.2
8	.779	10.3
9	.850	10.6
10	.899	11.1
11	.933	11.8
12	.956	12.5
13	.972	13.4
14	.982	14.3
15	.990	15.1

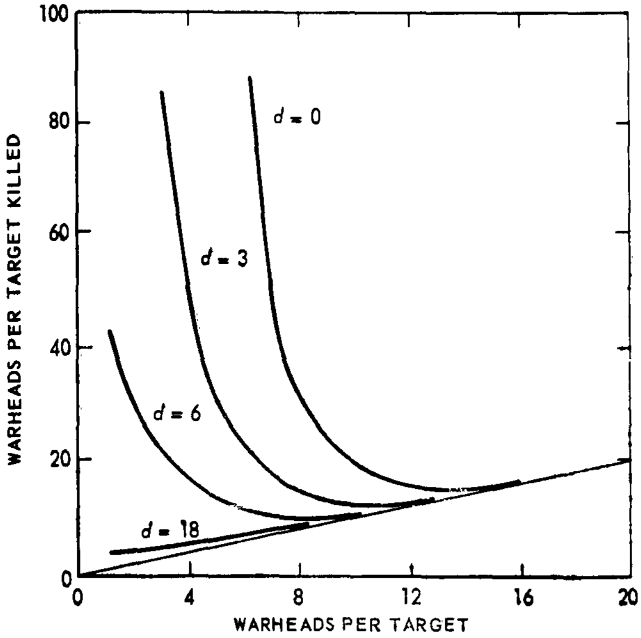

Fig. 3—Warheads Used as a Function of Warheads per Target Killed
$p_m = .75$; $c = 20$

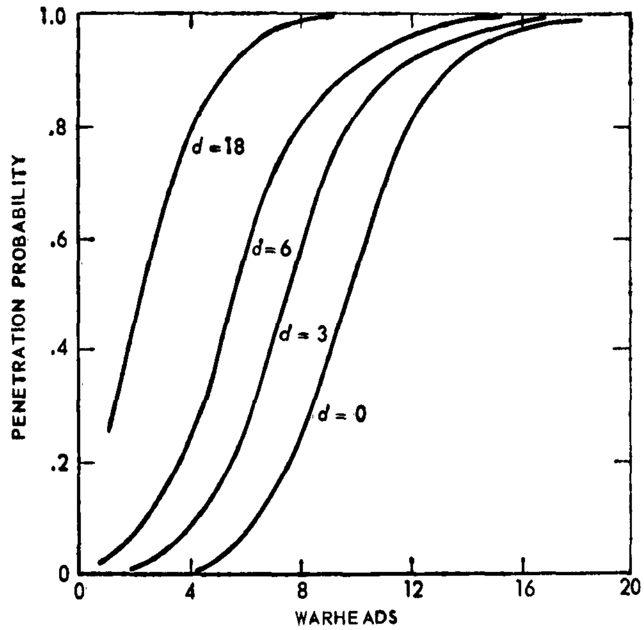

Fig. 4—Warhead Requirement as a Function of Penetration Probability
$p_m = .75$; $c = 20$

As a final illustration of the optimum attack, suppose the offense has a fixed number of warheads, e.g., 100. He will allocate these warheads to a set of enemy targets each defended by an AMM system of 20 missiles, each with SSKP = .75. Table 2 indicates that allocating 13 warheads without decoys to each target will result in the maximum expected number of targets destroyed. Further it can be expected that each of the targets attacked will be destroyed with probability .887. This assumes of course that at least one warhead will destroy the target. The offense can therefore expect to destroy $(100/13) \times (.887) = 6.82$ targets.

On the other hand, if the offense allocates only 7 warheads and 6 decoys per target (note that this is the same number of objects per target as in the previous example), it is evident from Table 4 that he can expect to destroy $(100/7)(.684) = 9.77$ targets, or an increase of 46 percent.

APPENDIXES

Appendix A

DEFENSIVE FIRE DOCTRINE

The model developed in this paper is based on the premise that the AMMs are deployed as uniformly as possible against the incoming objects; i.e., one missile is assigned to each incoming object until all objects are covered. Then, if doubling up is desired, another AMM is assigned to each object until each is covered twice, etc. A rigorous proof that a uniform fire doctrine is more effective than one deploying the missiles in a nonuniform manner would necessarily be untenable because of the infinite number of nonuniform cases; however, it seems intuitively that uniform firing would be much better than completely random firing, but not so clear is the case where just two AMMs are doubled up in order to increase the SSKP at the expense of letting an object (hopefully a decoy!) through free of charge.

In this appendix, two nonuniform deployments are considered and shown to be less effective than uniform deployment. In each example w warheads and d decoys arrive over the defended area, and the probability of at least one warhead penetrating is determined.

If the defense assigns one AMM to each incoming object, the penetration probability can be obtained from Eq 20 and is given by

$$P(1) = 1 - p_m^w \tag{A1}$$

Instead, if the defense is doubled on one object thereby leaving another object unattacked, the probability of penetration is obtained as follows: The SSKP of the double missile becomes $p_{2m} = 1-(1-p_m)^2$, and the probability that it attacks a decoy is $d/(w+d)$. The probability that this "new" missile attacks a warhead is $w/(w+d)$. The probability that at least one warhead penetrates is obtained using Eq 20 and is given by

$$\begin{aligned} P(1) &= \frac{d}{w+d}\left\{1-\frac{d-1}{w+d-1}\,p_m^w\right\}+\frac{w}{w+d}\left\{p_{2m}\left(1-\frac{d}{w+d-1}\,p_m^{w-1}\right)+1-p_{2m}\right\} \\ &= 1-\frac{d}{(w+d)(w+d-1)}\,p_m^{w-1}\left\{(d-1)p_m+wp_{2m}\right\} \end{aligned} \tag{A2}$$

It is desired to show the probability obtained from Eq A1 is less than that obtained from Eq A2 or

$$p_m^w > \frac{d}{(w+d)(w+d-1)}\,p_m^{w-1}\left\{(d-1)p_m+wp_{2m}\right\}$$

and it can be shown with some algebraic manipulation that the above inequality is equivalent to

$$p_m > 1 - \frac{w+d-1}{d}$$

which is valid for all w, $d \geq 1$.

Next suppose the defense has covered all incoming objects once and has two extra AMMs, i.e., $c = w+d+2$. The following argument demonstrates that these two extra AMMs should be paired with two of the assigned missiles and not tripled with one of the assigned missiles. The penetration probability is obtained from Eq 11 and, when the extra missiles are paired, is given by

$$P(1) = 1 - \frac{1}{\binom{w+d}{2}} \sum_{k=0}^{w} \binom{w}{k} \binom{d}{w+d-2-k} p_m^k \, p_{2m}^{w-k} \tag{A3}$$

while the probability of penetration when the missiles are tripled is given by

$$P(1) = 1 - \frac{1}{\binom{w+d}{1}} \sum_{k=0}^{w} \binom{w}{k} \binom{d}{w+d-1-k} p_m^k \, p_{3m}^{w-k} \tag{A4}$$

It is desired to show that the probability of penetration obtained from Eq A3 is less than that obtained from Eq A4 or

$$\frac{1}{\binom{w+d}{2}} \sum_{k=0}^{w} \binom{w}{k} \binom{d}{w+d-2-k} p_m^k \, p_{2m}^{w-k} > \frac{1}{\binom{w+d}{1}} \cdot \sum_{k=0}^{w} \binom{w}{k} \binom{d}{w+d-1-k} p_m^k \, p_{3m}^{w-k}$$

It can be shown with tedious algebra that the above inequality is equivalent to

$$p_m < 1$$

which is always valid.

Appendix B

A FORTRAN PROGRAM OF THE MATHEMATICAL MODEL

This appendix includes the FORTRAN program of Eq 24 shown in Fig. B1 and the flow diagram for this statement (Fig. B2).

Input

A. Table
5 cards containing the Log Factorial Table of n, $1 \le n \le 50$.

B. Data cards Format

Columns	Variable input[a]
1–4	c
5–8	w
9–12	d
13–16	p_m
17–20	u
24	1 (on last card only)

[a]All inputs except p_m are integers. p_m has an assumed decimal point between columns 14 and 15.

Output

$u \le 10$ because of output format.

```
*       XEQ
*       PRINT A 3  PUNCH B 4
*       LIST
*       LABEL
CMM          PENETRATION PROBABILITIES       USING LOG FACTORIAL TABLE
        DIMENSION COEI(5),TABLE(50),P(11)
        ITAPE=5
        JTAPE=6
        WRITE OUTPUT TAPE JTAPE,81
        WRITE OUTPUT TAPE JTAPE,80
        WRITE OUTPUT TAPE JTAPE,82
        WRITE OUTPUT TAPE JTAPE,83
        M1=1
        N1=10
        DO 2 IA=1,5
        READ INPUT TAPE ITAPE,90,(TABLE(I),I=M1,N1)
        M1=M1+10
      2 N1=N1+10
      1 READ INPUT TAPE ITAPE,59,C,W,D,PM,MU,NUMV
        DO 3 KP=1,11
      3 P(KP)=0.
        IC=C
        IWD=W+D
        I=IC/IWD
        AI=I
        COMPC=C-(AI-1.)*(W+D)
        PIM= 1.-(1.-PM)**I
        PIM1=1.-(1.-PM)**(I+1)
        XI=2.*(W+D)-COMPC
        SUMK=0.
        DO 71 K1=1,MU
        K=K1-1
        IW=W
        SUMR=0.
        IW1=IW+1
        DO 60 IR1=1,IW1
        IR=IR1-1
        SUMJ=0.
        DO 31 J1=1,IR1
        J=J1-1
        IC=1
        L=IR
        M=J
        GO TO 99
     20 IC=2
        L=IW-IR
        M=IW-K-J
        GO TO 99
     30 IF(J)73,301,303
    301 IF(PIM)73,302,303
    302 SUMJ=SUMJ+(COEI(1)*        (1.-PIM)**(IR-J)*COEI(2)*PIM1**(IW- K-J)
       1*(1.-PIM1)**( K-IR+J))
```

Fig. B1—FORTRAN Statement of Eq 24

```
      GO TO 31
  303 SUMJ=SUMJ+(COEI(1)*PIM**J*(1.-PIM)**(IR-J)*COEI(2)*PIM1**(IW- K-J)
     1*(1.-PIM1)**( K-IR+J))
   31 CONTINUE
      IC=3
      L=IW
      M=IR
      GO TO 99
   40 IC=4
      L=D
      IX=XI
      M=IX-IR
      GO TO 99
   50 IC=5
      L=W+D
      M=XI
      GO TO 99
   60 SUMR=SUMR+((COEI(3)*COEI(4)/COEI(5))*SUMJ)
      KP=K1
      P(KP)=SUMR
   71 SUMK=SUMK+SUMR
      ANS=1.-SUMK
      WRITE OUTPUT TAPE JTAPE,84,C,W,D,PM,MU,(P(KP),KP=1,11),ANS
      IF(NUMV-1)1,73,73
   73 CALL EXIT
   99 COEI(IC)=1.
      IF(M)103,100,102
  100 IF(L)103,150,150
  102 IF(L-M)103,150,105
  103 COEI(IC)=0.
      GO TO 150
  105 IF(L-50)106,106,200
  106 IF(M-50)107,107,200
  107 LM=L-M
      ARG=2.30259*(TABLE(L)-TABLE(M)-TABLE(LM))
      COEI(IC)=EXPF(ARG)
  150 GO TO (20,30,40,50,60),IC
  200 WRITE OUTPUT TAPE JTAPE,85
      GO TO 73
   59 FORMAT (3(F4.0),F4.2,2(I4))
   80 FORMAT(50X,26H PENETRATION PROBABILITIES///)
   81 FORMAT(1H1)
   82 FORMAT(15X,6H INPUT,48X,7H OUTPUT///)
  830FORMAT(1X,2H C,4X,2H W,4X,2H D,4X,3H PM,3X,3H MU,10X,3H P0,3X,3H P
    11,3X,3H P2,3X,3H P3,3X,3H P4,3X,3H P5,3X,3H P6,3X,3H P7,3X,3H P8,3
    2X,3H P9,3X,4H P10,7X,6H P(MU)///)
   84 FORMAT(1X,F3.0,3X,F3.0,3X,F3.0,3X,F3.2,3X,I3,9X,11(1X,F4.3,1X),6X,
    1F6.3///)
   85 FORMAT (15H L OR M TOO BIG)
   90 FORMAT(10F7.5)
      END
*     DATA
```

Fig. B1 (continued)

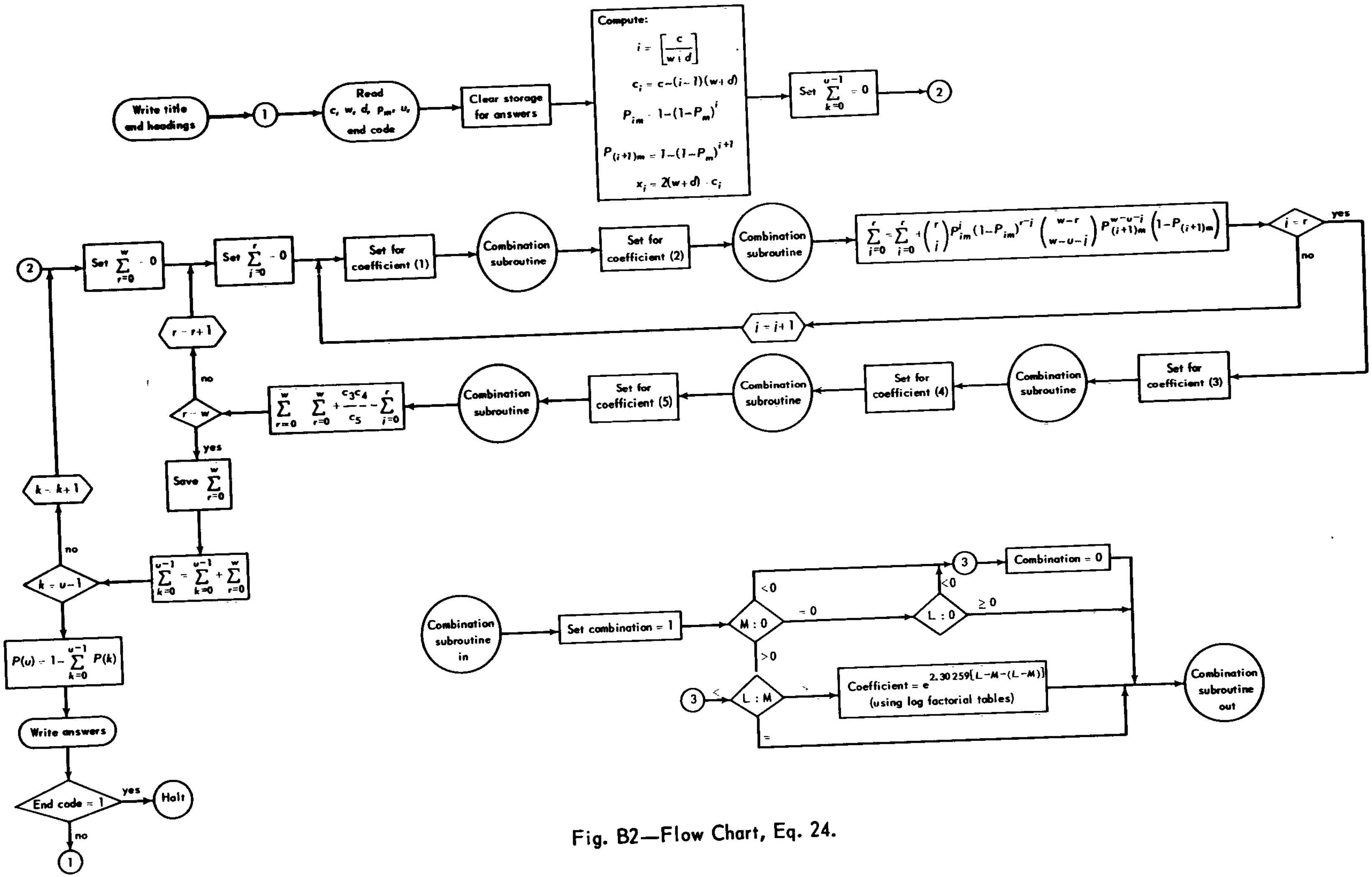

Fig. B2—Flow Chart, Eq. 24.

REFERENCES

William Feller, An Introduction to Probability Theory and Its Applications, John Wiley & Sons, Inc., New York, 1957.

Thornton Read, "Strategy for Active Defense," unpublished material.